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MULTISTAGE LOT-SIZING: AN ITERATIVE PROCEDURE.(U)

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MULTISTAGE LOT-SIZING: AN ITERATIVE PROCEDURE

by

STEPHEN C. GRAVES

Technical Report No. 164

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FOREWORD

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ABSTRACT

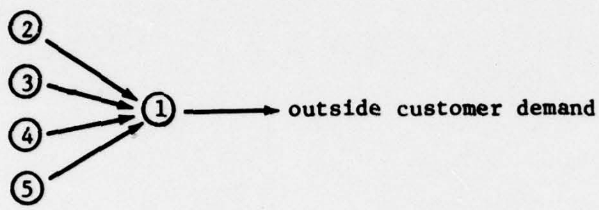
This paper considers the lot-sizing problem in a multistage inventory system. External demand may occur at any stage, and is assumed to be known over a finite horizon. A heuristic iterative procedure is proposed and tested for finding a periodic review schedule to minimize inventory and setup costs.

1. Introduction

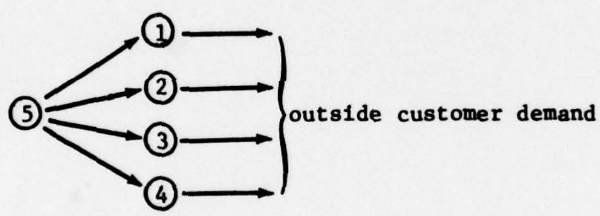
In this paper we consider a dynamic lot-sizing problem for a multistage inventory system. The lot-sizing problem is to determine replenishment quantities for an inventory system so as to satisfy all demand requirements at minimum system cost. A multistage inventory system is a connected set of stages representing the steps for assembly and/or distribution for a series of products. Typically we may characterize a multistage inventory system as an acyclic network modified to include links between the stages and the outside customer demand; examples of such systems are given in Figure 1. A standard classification is to distinguish between assembly and arborescent inventory systems; for an assembly system (Figure 1b), each stage either has a unique immediate successor or satisfies directly outside customer demand, while for an arborescent system (Figure 1c), each stage has either a unique immediate predecessor or no predecessors at all. A serial system (Figure 1a) is both an arborescent and an assembly system, while a general multistage system need not be either an arborescent or an assembly system.

The objective of this paper is to propose and test a heuristic lot-sizing procedure for a general multistage discrete-time inventory system. We assume that outside customer demand is known by period over a finite horizon, and must be satisfied from on-hand inventory with backordering not allowed. All lead times for supplying a stage from an immediate predecessor are given, and without loss of generality are assumed to be zero. A lot-sizing procedure specifies a periodic schedule indicating the size and timing of inventory replenishments so that each stage satisfies the demand placed upon it by succeeding stages. The optimality criterion is to minimize total cost over the finite horizon where there are two types of costs

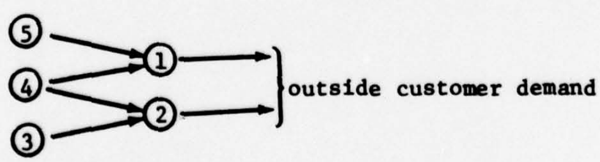
(a) 5 stage serial system



(b) 5 stage assembly system



(c) 5 stage arborescent system



(d) 5 stage general system

Figure 1: Examples of Multistage Inventory Systems

at each stage: a fixed cost for placing an order, and an inventory holding cost which we assume to be proportional to the end-of-period inventory at each stage.

There has been a great amount of work done on dynamic lot-sizing problems, beginning with the classical work of Wagner and Whitin [15] for the single-stage problem. Significant extensions have been made by Wagner [14] to include dynamic cost functions, by Zabel [16], Eppen, et. al. [5], Kunreuther and Morton [7],[8], Blackburn and Kunreuther [2], and Lundin and Morton [10] to extend and generalize the planning horizon theorems, and by Zangwill [17] to include backlogging of demand.

For the dynamic multistage lot-sizing problem, Zangwill [17] has shown that for a series system the optimal solution is contained in the set of extreme flows of a single-source network, and has given a dynamic programming recursion for calculating the optimal policies. Love [9] also considers a series inventory problem, and gives an alternate algorithm, which exploits the nested property of the solution.

Veinott [13], Crowston and Wagner [4], and Kalymon [6] have considered more general multistage problems, and have given algorithms for solving specific problems; however in all three papers, the algorithms are quite complex, with the amount of work required increasing exponentially with the number of stages, or with the number of time periods, or with both.

Due to the complexity of the general multistage problem, many heuristic procedures have been proposed. The most common form of heuristic is to consider the stages sequentially, starting with the lowest echelon stages[1], and scheduling each stage with a single-stage procedure which may by itself be a heuristic. Examples of such heuristics are given by McLaren [11],

---

[1] The lowest echelon stages are those nearest to the customer.

McLaren and Whybark [12], Biggs, et. al. [1], and Blackburn and Millen [3]. For an N-stage system, the amount of work necessary for these heuristics, which we term as single-pass heuristics, is comparable to that needed for solving N single-stage problems. All of the reported work has been restricted to assembly systems. From this work, a heuristic given in [11], [12] seems to give the best performance, although any conclusions are limited by the scope of the computational studies; this heuristic is discussed in greater detail in Section 3.

The intent of this paper is to present and test a new type of heuristic, a multipass heuristic for the multistage lot-sizing problem. Whereas a single-pass heuristic sequentially schedules each stage and then stops, a multipass heuristic does not stop after the "single-pass", but continues to revise the current schedule in an iterative fashion until no further improvements in the schedule are possible. Admittedly, the multipass heuristics require more computational effort than single-pass heuristics, but hopefully much less than an optimal algorithm. The schedule performance of a multipass heuristic should likewise be bounded by that of a single-pass heuristic and an optimal algorithm.

The remainder of the paper is organized as follows: In the next section, a multipass heuristic is developed and is shown to be monotonic and convergent. In Section 3, we report and discuss our computational experiments comparing the multipass heuristic with two single-pass heuristics, and with an optimal algorithm.

2. The Multipass Heuristic

The multipass heuristic consists of two phases, one in which the current schedule is revised and the other in which various stages are "collapsed" into other stages based on the current schedule. The presentation in this section will first develop the logic for revision and then will consider the collapsing procedure; this development is motivated first by considering simple systems and then is generalized for more complex systems.

2.1 A Revision Procedure for a Two-Stage System

Consider a two-stage system as depicted in Figure 2. Item 2 is the single component for item 1, the final product. For each unit of item 1 assembled, β units of item 2 are required. This two-stage structure is the simplest multistage system, and is used here in order to simplify the presentation of the proposed heuristic. All results for this system can be generalized to more complex multistage structures.

$$2 \xrightarrow{\beta} 1 \longrightarrow \text{customer demand}$$

Figure 2: A Two-Stage System

Define the following:

d_{it} = external demand for item i in time period t,

S_i = setup cost for item i,

h_i = holding cost for item i,

v_{it} = variable production cost for item i in time period t,

T = length of scheduling horizon in periods.

We assume that the initial inventory for both items is zero, the production lead-time is zero, and $d_{11} > 0$. Also, note that item 2 may have external

demand, independent of the demand imposed on it by item 1. For notational convenience, we will identify the single-stage problem with demand requirements d_t for $t=1,\ldots,T$, setup cost S, holding cost h, and variable production cost v_t for $t=1,\ldots,T$, as $WW[d_t,S,h,v_t]$, i.e. the Wagner-Whitin problem. Now the initial heuristic (H1) is as follows:

(H1):

1. Solve $WW[d_{1t},S_1,h_1,v_{1t}]$. If x_{1t} is the production quantity of item 1 in time period t, then $\beta\, x_{1t}$ is the demand imposed on item 2 by the production schedule of item 1.
2. Set $\hat{d}_{2t} = d_{2t} + \beta x_{1t}$ = total demand for item 2 in period t. Solve $WW[\hat{d}_{2t},S_2,h_2,v_{2t}]$. From this solution, determine γ_{2t} = marginal cost of increasing demand for item 2 in period t by one unit.
3. Set $\hat{v}_{1t} = v_{1t} + \beta\gamma_{2t}$ = restated variable production cost in period t. Solve $WW[d_{1t},S_1,h_1,\hat{v}_{1t}]$. If there is no change in the schedule for item 1, then stop; if there is a change, then return to step 2.

This procedure is an iterative procedure in which two single-stage problems are solved at each iteration. We will show that the procedure is nonincreasing in cost, and hence, convergent. First, though, we indicate how the marginal costs γ_{2t} are determined.

Define γ_{2t} to be the marginal cost of increasing by one unit the demand requirements of item 2 in time period t given a production schedule for item 2; γ_{2t} can be interpretted as a shadow price on demand for a fixed schedule for item 2. Let τ be the last time period prior to t in which there is production of item 2 (i.e. the setup cost is incurred). That is, if x_{2t} is production in time period t, then

$$x_{2\tau} > 0$$

$$x_{2j} = 0 \qquad \text{for } j=\tau+1,\ldots,t,$$

or $\quad x_{2t} > 0$ and $t = \tau$.

Now we specify γ_{2t} as

$$\gamma_{2t} = v_{2\tau} + (t-\tau)\cdot h_2$$

That is, if demand in period t is increased by one unit, production in period τ ($x_{2\tau}$) increases by one unit with an incremental cost consisting of the variable production cost in period τ, $v_{2\tau}$, plus the holding cost from period τ to t which is $(t-\tau)\cdot h$. Note that we assume τ always exists; this is true provided that there is no initial inventory and demand in the first period (external or induced) is nonzero.

We now show that the procedure coverges. Define the following:

c_{in} = cost (using <u>original</u> costs S_i, h_i, v_{it}) for schedule of item i after the n^{th} iteration,

$\{x^n_{1t}\}$ = schedule for item i after n^{th} iteration,

$\{\gamma^n_{2t}\}$ = marginal costs generated after n^{th} iteration from $\{x^n_{2t}\}$,

$\Delta x_t = x^{n+1}_{1t} - x^n_{1t}$ = change in schedule for item 1 in period t.

At the $n+1^{st}$ iteration, for item 1 we solve the "revised" problem $WW[d_{1t}, S_1, h_1, \hat{v}_{1t}]$, where $\hat{v}_{1t} = v_{1t} + \beta\gamma^n_{2t}$, to obtain the schedule $\{x^{n+1}_{1t}\}$. If the cost for schedule $\{x^{n+1}_{1t}\}$ using the actual production costs v_{1t} is $c_{1,n+1}$, then the "revised" cost using $\hat{v}_{1t}$ is $c_{1,n+1} + \sum_{t=1}^{T} \gamma^n_{2t}(\beta x^{n+1}_{1t})$. Since $\{x^{n+1}_{1t}\}$ is the optimal schedule for the "revised" problem $WW[d_{1t}, S_1, h_1, \hat{v}_{1t}]$, we have

$$(1) \qquad c_{1,n+1} + \sum_{t=1}^{T} \gamma^n_{2t}(\beta x^{n+1}_{1t}) \leq c_{1n} + \sum_{t=1}^{T} \gamma^n_{2t}(\beta x^n_{1t})$$

where the RHS is the cost for the "revised" problem using the schedule $\{x_{1t}^n\}$. We can rewrite (1) as

$$c_{1,n+1} \le c_{1,n} - \sum_{t=1}^{T} \gamma_{2t}^n (\beta\Delta x_t) \tag{2}$$

For item 2 at the n+1st iteration, the demand is given by $\hat{d}_{2t}^{n+1} = d_{2t} + \beta x_{1t}^{n+1}$. The previous schedule $\{x_{2t}^n\}$ may be modified to satisfy this demand $\{\hat{d}_{2t}^{n+1}\}$ by just adjusting the positive order quantities (i.e., $x_{2t}^n > 0$); if $\{y_t\}$ is the modified schedule, we have

$$y_t = 0 \qquad \text{if } x_{2t}^n = 0,$$

and

$$\begin{aligned} y_t &= x_{2t}^n + \sum_{j=t}^{\tau-1} \left(\hat{d}_{2j}^{n+1} - \hat{d}_{2j}^n\right) \\ &= x_{2t}^n + \beta \sum_{j=t}^{\tau-1} \Delta x_j \qquad \text{if } x_{2t}^n > 0, \end{aligned}$$

where τ is defined such that $x_{2j}^n = 0$ for $j=t+1,\ldots,\tau-1$, and $x_{2\tau}^n > 0$, or $\tau = T+1$. The cost for item 2 for the schedule $\{y_t\}$ can be shown to be equal to $c_{2n} + \beta \sum_{t=1}^{T} \gamma_{2t}^n(\Delta x_t)$; for $c_{2,n+1}$ being the cost for the optimal schedule, we have by definition

$$c_{2,n+1} \le c_{2n} + \beta \sum_{t=1}^{T} \gamma_{2t}^n (\Delta x_t). \tag{3}$$

Adding (2) and (3), we obtain

$$c_{1,n+1} + c_{2,n+1} \le c_{1,n} + c_{2,n}\,. \tag{4}$$

Thus at each iteration, total cost is nonincreasing. If we ignore the possibility of cycling, the procedure converges since the single-stage (WW) problem considers only extreme point schedules, of which there are a finite number. Nevertheless, it may be possible for the solution to cycle; that is, the procedure cycles over a finite number of solutions with no cost improvement.

This may be avoided by stopping the procedure if any solution is repeated, or by perturbing the data so that no two solutions may have the same cost. One possible perturbation is to redefine the setup cost for stage 1 in time period t as $S_{1t} = S_1 + \varepsilon^t$ where ε is a small positive constant.

2.2 Extension to More Complex Structures

The heuristic (H1) and its convergence properties are extendable to more complex structures. For instance consider the following two-echelon, 6-stage system (Figure 3):

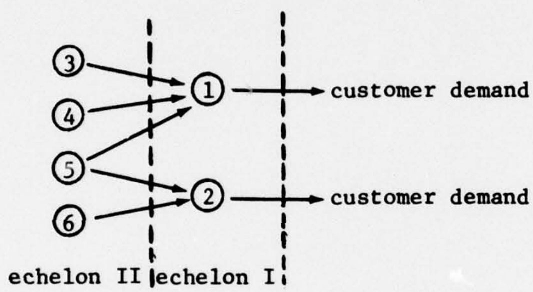


Figure 3: Six-Stage, Two-Echelon System

Define β_{ij} to be the number of units of item i needed for one unit of item j. Then (H1) is modified as follows:

1. In step 1, the independent single-stage (WW) problems for the first echelon (items 1 and 2) are solved; these solutions determine the demand for echelon II items.
2. In step 2, the independent WW problems for the second echelon (items 3, 4, 5, and 6) are solved. Note that for item 5, we have $\hat{d}_{5t} = d_{5t} + \beta_{51}x_{1t} + \beta_{52}x_{2t}$ where $\beta_{5i}x_{it}$ is the induced demand from the schedule for item i, i=1,2. For each item in echelon II, the marginal costs γ_{it} can be computed as before.

3. Step 3 is identical to step 1 except that the "revised" WW problems are solved for items 1 and 2. Here the variable production costs are adjusted to reflect the effect of the higher echelon. For instance, we have $\hat{v}_{1t} = v_{1t} + \beta_{31}\gamma_{3t} + \beta_{41}\gamma_{4t} + \beta_{51}\gamma_{5t}$.

The proof of the convergence of this procedure is identical in structure to that given in the previous section.

The extension of the heuristic to more than two echelons is slightly more complex. The original heuristic is inherently geared for iterating between two echelons. With more than two echelons, we must decide in what order the procedure cycles over the echelons. This can be best illustrated by considering the following three-echelon, three-stage system (Figure 4):

$3 \longrightarrow 2 \longrightarrow 1 \longrightarrow$ customer demand

Figure 4: Three-Stage Serial System

Here each stage corresponds to an echelon.

Again the revisions to (H1) are straight-forward for the structure. Demand at the higher echelons, given schedules for the lower echelons, is

$$\hat{d}_{2t} = d_{2t} + \beta_{21}x_{1t}$$

$$\hat{d}_{3t} = d_{3t} + \beta_{32}x_{2t}$$

The revised variable costs are computed as

$$\hat{v}_{1t} = v_{1t} + \beta_{21}\gamma_{2t}$$

where $$\gamma_{2t} = \hat{v}_{2\tau} + (t-\tau)h_2$$

and $$\hat{v}_{2\tau} = v_{2\tau} + \beta_{32}\gamma_{3\tau}$$

for τ being the last production period of item 2 prior to period t. Note that γ_{2t}, the marginal cost for item 2, reflects not only the incremental cost at stage 2, but also the incremental cost at stage 3.

Now, for this system two procedures suggest themselves for iterating over the echelons:

a) echelon I $\rightarrow$ II $\rightarrow$ III $\rightarrow$ I $\rightarrow$ II ...

Here a completely cyclic procedure is used; having scheduled the echelons from I to III, the procedure returns to echelon I and tries to improve the schedule.

b) echelon I $\rightarrow$ II $\rightarrow$ III $\rightarrow$ II $\rightarrow$ III ... (until II and III converge) $\rightarrow$ I $\rightarrow$ II $\rightarrow$ III $\rightarrow$ II $\rightarrow$ III ...

Here, given a schedule for echelon I, the two echelon problem consisting of II and III is completely solved. Given the solution to II and III, the procedure returns to I and revises its schedule. The procedure then repeats with the new schedule for I.

It is not clear which of these two procedures is more effective. Preliminary computational tests on small examples suggest that the schedules are relatively insensitive to the procedure, but that procedure b) requires more work than a).

Both procedures a) and b) can be shown to converge. The convergence proof for procedure a) is given in the Appendix. For procedure b) the proof follows directly that given for the two-stage system; to see this, note that this procedure may be viewed as the solution to a nested series of two-stage problems, each of which is well-behaved.

The M-echelon, N-stage problem ($M < N$) is now just a combination of the previous two extensions. Again, for $M > 2$, we have the problem of deciding how to iterate over the echelons.

2.3 Additional Improvements by Stage Collapsing

The heuristic (H1) is an improvement heuristic. Unfortunately examples can be easily constructed to show that it is not an optimal procedure. Indeed, it is possible for (H1) to give a schedule which not only is not optimal, but which can be easily improved. This section characterizes such situations and proposes a collapsing routine which is to be applied to the schedule generated by (H1).

Consider a two-stage system (i.e. Figure 2), and assume there is no external demand for item 2 ($d_{2t} = 0$). For this system it is quite possible for (H1) to give a schedule such that

$$x^n_{2t} = \beta x^n_{1t} \qquad \text{for } t=1,2,\ldots,T. \tag{5}$$

That is, item 2 is simultaneously produced whenever item 1 is produced, and no inventory is ever kept for item 2. Note that if $\{x^n_{1t}\}$ is the schedule for item 1 after the n^{th} iteration of (H1), and if $x^n_{2t} = \beta x^n_{1t}$, then we have $x^{n+1}_{1t} = x^n_{1t}$ which implies convergence for (H1). This will occur because at iteration n+1 the "revised" variable production cost of item 1 in product t is unchanged if $x^n_{1t} > 0$, but is higher if $x^n_{1t} = 0$. Consequently, the new schedule for item 2 does not alter the scheduling of item 1.

An implication from (5) is that the two-stage problem may be restated as a single-stage WW problem with setup cost $S = S_1 + S_2$, holding cost $h = h_1$, variable production cost $v_t = v_{1t} + \beta v_{2t}$, and demand $d_t = d_{1t}$. The otpimal solution of this single-stage problem must, by definition, be no worse than the schedule in (5). Thus, if (H1) generates a schedule satisfying (5), by collapsing stage 2 into stage 1 we can define a single-stage problem, from which an improved schedule may be found.

This improvement can be generalized for more complex N-stage problems.

We need restate condition (5) for stage i as

(6) $$x^n_{it} = \beta_{ij} x^n_{jt} \qquad \text{for } t=1,2,\ldots,T,$$

where stage j is the <u>unique</u> successor to stage i. Now stage i may be collapsed into stage j by redefining stage j to have setup cost $S = S_j + S_i$, holding cost $h = h_j$, and variable production cost $v_t = v_{jt} + \beta_{ij} v_{it}$. Note that the immediate predecessors or inputs to stage i are now direct inputs to stage j. After collapsing as many stages as possible, we may now reapply heuristic (H1) to find an improved schedule; the procedure stops when a schedule is found which cannot be improved by (H1), and from which no more stages can be collapsed.

3. Computation Tests

In order to test the performance of the proposed multipass heuristic, we tested the heuristic procedure on 50 test problems generated for each of five multistage assembly systems. Figure 5 gives the five assembly systems each with five stages; these systems are the same as used in [11], [12].[1] In generating the test problems, we normalized $h_5 = 1$; for $j=1,\ldots,4$ we set $h_j = e + \sum_{i\varepsilon B(j)} h_i$ where $B(j)$ is the set of immediate successors to j and e is selected from a uniform distribution with values $e = 0.1, 0.5, 1.0,$ and 2.0. We selected S_j from a uniform distribution with values $S_j = 150,$ $300, 600, 1500$; we selected d_{1t} from a uniform distribution with values $d_{1t} = 0, 10, 20, 30, 40, 100, 200, 400$. There was no other external demand ($d_{it} = 0$ for all t, and $i=2,\ldots,5$), all variable production costs v_{it} were zero, and the length of the horizon T was 12 periods.

For each test problem, we compared the multipass heuristic with two single-pass heuristics and with an optimal algorithm. The mutlipass heuristic (MP) that was implemented for these test problems iterated in a full cyclic fashion over the stages; that is, the procedure scheduled stages $1,2,\ldots,5$ using a Wagner-Whitin algorithm, and then returned to stage 1 to begin the revisions. Alternative schemes were examined, but did not seem to give significant improvements. The optimal schedule was found using the dynamic programming procedure of Crowston and Wagner [4]. One single-pass heuristic (SP-WW) was to schedule each stage using the Wagner-Whitin algorithm. The second single-pass heuristic (SP-MW) was that proposed

[1] Although all the test problems are for assembly systems, the multipass heuristic can be used for any multistage system. Assembly systems were chosen for the computational tests, because the multipass heuristic is to be compared with a single-pass heuristic proposed in [12] and with an optimal algorithm from [4], both of which <u>are</u> restricted to assembly systems.

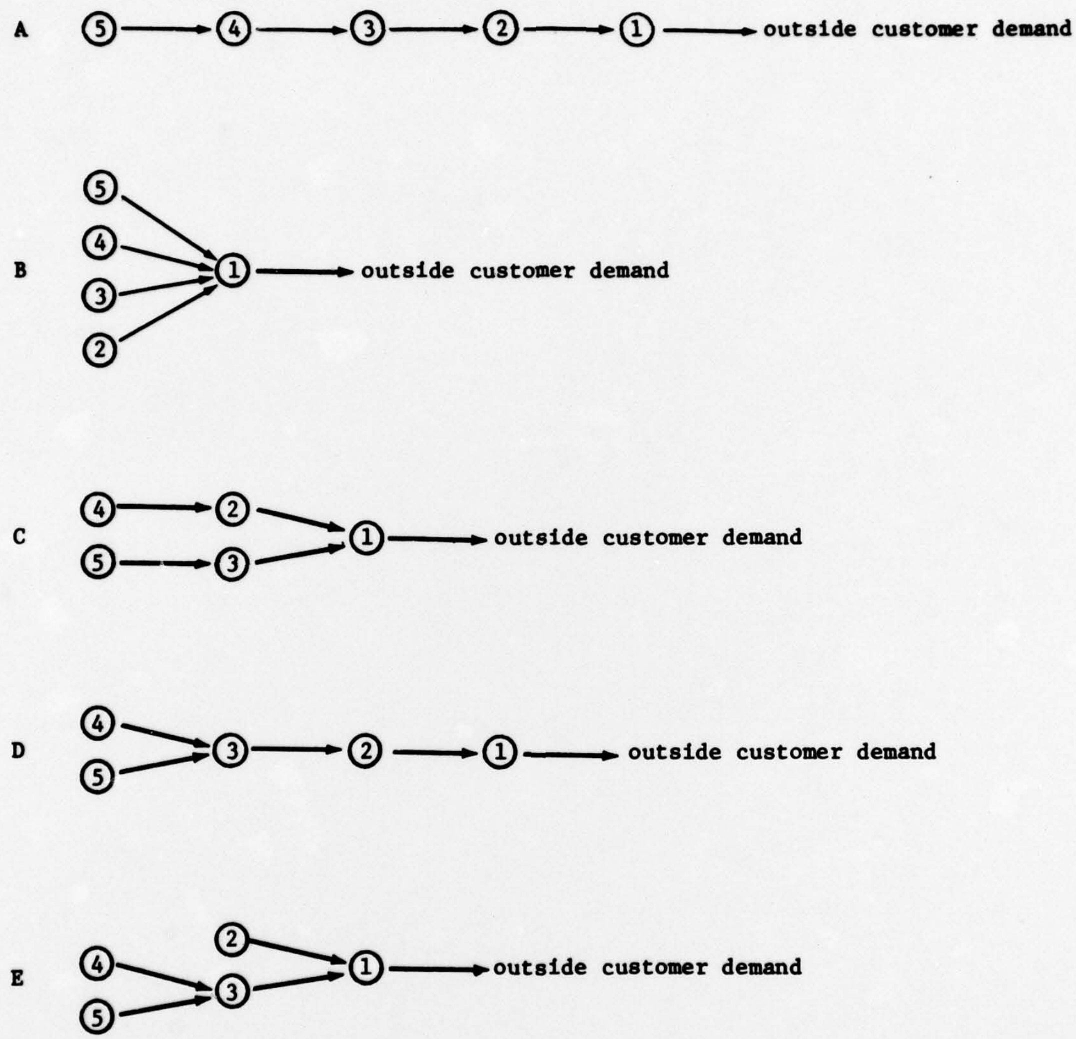


Figure 5: Multistage Assembly Systems

by McLaren and Whybark [12]; here each stage is scheduled using the Wagner-Whitin algorithm, but with the stage's setup cost inflated to reflect the possible setups for preceding stages.

The primary results are given in Tables 1 and 2; Table 1 reports the number of times each heuristic obtained the optimal solution, while Table 2 gives the average percentage cost penalty for the heuristics over the optimal procedure. From these tables, in the aggregate the SP-WW is dominated by the SP-MW which is dominated by the MP. However, for each problem structure there were exceptions to this domination such that on a few test problems SP-WW outperformed SP-MW, or SP-MW outperformed MP; SP-WW never did better than MP, since the multipass heuristic is an improvement routine which is initiated with the SP-WW schedule. We should note that even though the MP is optimal in about 90% of the test problems, compared with a rate of about 60% for SP-MW, both heuristics have minuscule average percentage cost errors. Also, the performance of both heuristics seemed to vary with the number of echelons in the system; the performance relative to the optimal procedure becomes poorer as the number of echelons grows.

The computational effort required for the heuristics and for the optimal procedure was consistent with their respective performances. Both the SP-WW and the SP-MW heuristics solve one WW problem per stage, or five WW problems for each test problem; on a PRIME 400 minicomputer, the 250 test problems using the SP-MW heuristic were solved in 53 seconds of cpu time (0.21 seconds/problem). The multipass heuristic due to its iterative nature, must solve a variable number of WW problems. It can be shown that a gross upper bound on the number of WW problems considered by the multipass heuristic, as implemented in this work, is $T \cdot N$ where T is the length of the horizon and N is the number of stages; since the WW problem can be solved

Table 1: Frequency of Optimal Schedules (out of 50 problems)

PROBLEM STRUCTURE	HEURISTIC: SP-WW	SP-MW	MP
A	2	19	41
B	4	42	48
C	2	27	44
D	2	20	46
E	4	31	47

Table 2: Average Percentage Cost Errors

PROBLEM STRUCTURE	HEURISTIC: SP-WW	SP-MW	MP
A	8.64%	2.10%	0.58%
B	3.27%	0.06%	0.05%
C	5.12%	0.64%	0.26%
D	7.45%	1.70%	0.31%
E	4.76%	0.61%	0.26%

efficiently, the multipass heuristic is an efficient procedure. For the 250 test problems, the average number of WW problems considered was slightly over 18 per test problem; the total cpu time for the 250 test problems was 115 seconds (0.46 seconds/problem). The optimal algorithm as reported in [4], is an inefficient procedure, in that the computational work grows exponentially with the length of the horizon; for the 250 test problems the optimal algorithm required 2077 seconds of cpu time (8.31 seconds/problem).

In summary, the multipass heuristic performs better than the leading single-pass heuristic, SP-MW, but requires more work. In light of the computational experience, we conjecture that the effort needed for the multipass heuristic will be proportional to that needed for a single-pass heuristic. Finally, we note that whereas the multipass heuristic is applicable to general multistage systems, both the SP-MW heuristic and the optimal algorithm are designed for assembly systems.

APPENDIX: Proof of Convergence for Three-Stage Serial System

Consider the three-stage system depicted in Figure 4. Assume that the revision procedure as outlined in Section 2.2 has been applied where the iteration over the echelons is in a pure cyclic fashion (e.g. echelon I → II → III → I → II ...). To show that the procedure converges, we will follow the arguments given in Section 2.1 for the two-stage system. For notational ease we assume that $\beta_{32} = \beta_{21} = 1$. After the n+1[st] iteration, for $\{x_{1t}^{n+1}\}$ being the schedule for item 1 with actual cost $c_{1,n+1}$, the "revised" cost using $\hat{v}_{1t}$ is given by $c_{1,n+1} + \sum_{t=1}^{T} \gamma_{2t}^{n} x_{1t}^{n+1}$. Since $\{x_{1t}^{n+1}\}$ is optimal for the "revised" problem, we have

(A1) $$c_{1,n+1} + \sum_{t=1}^{T} \gamma_{2t}^{n} x_{1t}^{n+1} \leq c_{1n} + \sum_{t=1}^{T} \gamma_{2t}^{n} x_{1t}^{n}$$

or

(A2) $$c_{1,n+1} \leq c_{1n} - \sum_{t=1}^{T} \gamma_{2t}^{n} (\Delta x_{1t})$$

where $\Delta x_{1t} = x_{1t}^{n+1} - x_{1t}^{n}$.

For item 2 at the n+1[st] iteration, the demand is given by $\hat{d}_{2t}^{n+1} = d_{2t} + x_{1t}^{n+1}$. A feasible schedule $\{y_t\}$ may be derived from the previous schedule $\{x_{2t}^{n}\}$ as follows:

$$y_t = 0 \quad \text{if} \quad x_{2t}^{n} = 0,$$

and $$y_t = x_{2t}^{n} + \sum_{j=t}^{\tau-1} \Delta x_{1j} \quad \text{if} \quad x_{2t}^{n} > 0$$

where τ is defined as the earliest period after t such that $x_{2\tau}^{n} > 0$. The cost associated with $\{y_t\}$ for the "revised" problem $WW[\hat{d}_{2t}, S_2, h_2, \hat{v}_{2t}]$ is

(A3) $$c_{2n} + \sum_{t=1}^{T} \gamma_{2t}^{n} (\Delta x_{1t}) + \sum_{t=1}^{T} \gamma_{3t}^{n} x_{2t}^{n}$$

where c_{2n} is the actual cost associated with schedule $\{x^n_{2t}\}$. To verify (A3), note that $c_{2n} + \sum_{t=1}^{T} \gamma^n_{3t} x^n_{2t}$ is the "revised" cost for schedule $\{x^n_{2t}\}$ assuming demand $\hat{d}^n_{2t} = d_{2t} + x^n_{1t}$. When the item's demand is restated as $\hat{d}^{n+1}_{2t} = \hat{d}^n_{2t} + \Delta x_{1t}$, schedule $\{x^n_{2t}\}$ need not be feasible; the feasible schedule $\{y_t\}$ dervied from $\{x^n_{2t}\}$, has additional costs which are given by $\sum_{t=1}^{T} \gamma^n_{2t}(\Delta x_{1t})$, by definition of the marginal costs.

The cost for the optimal schedule $\{x^{n+1}_{2t}\}$ for the "revised" problem $WW[\hat{d}_{2t}, S_2, h_2, \hat{v}_{2t}]$ is

$$\text{(A4)} \qquad c_{2,n+1} + \sum_{t=1}^{T} \gamma^n_{3t} x^{n+1}_{2t}$$

where $c_{2,n+1}$ is the actual cost of the schedule. Due to the optimality of $\{x^{n+1}_{2t}\}$ we have that (A4) is less than or equal to (A3), which may be written as

$$\text{(A5)} \qquad c_{2,n+1} \leq c_{2n} + \sum_{t=1}^{T} \gamma^n_{2t}(\Delta x_{1t}) - \sum_{t=1}^{T} \gamma^n_{3t}(\Delta x_{2t}).$$

For item 3 at the n+1[st] iteration, the demand is given by $\hat{d}^{n+1}_{3t} = d_{3t} + x^{n+1}_{2t}$. The previous schedule $\{x^n_{3t}\}$ may be modified to satisfy this demand in a similar fashion to the derivation of $\{y_t\}$ for item 2. The actual cost for this schedule can be shown to be equal to $c_{3n} + \sum_{t=1}^{T} \gamma^n_{3t}(\Delta x_{2t})$. If schedule $\{x^{n+1}_{3t}\}$ with cost $c_{3,n+1}$ is the optimal schedule to $WW[\hat{d}_{3t}, S_3, h_3, v_{3t}]$, then we must have

$$\text{(A6)} \qquad c_{3,n+1} \leq c_{3n} + \sum_{t=1}^{T} \gamma^n_{3t}(\Delta x_{2t})$$

due to the optimality of $\{x^{n+1}_{3t}\}$.

By combining the inequalities (A2), (A5), and (A6), we have

$$c_{1,n+1} + c_{2,n+1} + c_{3,n+1} \leq c_{1n} + c_{2n} + c_{3n},$$

which states that at each iteration total cost is nonincreasing. By similar reasoning to that given for the two-stage example, this result is sufficient to guarantee convergence.

To extend this result to an N-stage serial system is straightforward. Inequalities (A2) and (A6) can be shown true for stage (item) 1 and for stage (item) N, respectively. For stages 2,3,...,N-1, an inequality identical to (A5) can be established. By combining these inequalities the desired result is obtained.

For more general multistage systems, the convergence proof is similar in structure but is complicated by the notational needs. Essentially, for each echelon in the system an inequality similar to (A2), (A5), or (A6) is derived which relates the total actual costs of the echelon at the $n+1^{st}$ iteration to those costs at the n^{th} iteration. The details of this proof have been omitted.

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